Size of quantum networks

Ginestra Bianconi
Département de Physique Théorique, Université de Fribourg Pérolles, CH-1700 Fribourg, Switzerland
(Received 23 January 2003; published 23 May 2003)

The metric structure of bosonic scale-free networks and fermionic Cayley-tree networks is analyzed, focusing on the directed distance of nodes from the origin. The topology of the networks strongly depends on the dynamical parameter T, called the temperature. At $T=\infty$ we show analytically that the two networks have a similar behavior: the distance of a generic node from the origin of the network scales as the logarithm of the number of nodes in the network. At $T=0$ the two networks have an opposite behavior: the bosonic network remains very clusterized (the distance from the origin remains constant as the network increases the number of nodes), while the fermionic network grows following a single branch of the tree, and the distance from the origin varies as a power law of the number of nodes in the network.

DOI: 10.1103/PhysRevE.67.056119 PACS number(s): 89.75.Hc

I. INTRODUCTION

Complex networks representing systems of interacting units have been studied and classified [1,2] according to their different geometrical and topological properties. Among complex networks, scale-free networks, with a power-law connectivity distribution, have been found to describe different systems of nature and society [3–5]. Recently, several models [3–6] have been formulated that generate such structures, the prototype of them being the Barabási-Albert (BA) model [7]. Scale-free networks are particularly interesting because their highly inhomogeneous structure induces peculiar effects in the dynamical models that can be defined on them, such as the absence of percolation [8] and an epidemic threshold [9], the infinite Curie temperature for the paramagnetic to ferromagnetic transition in the frame of the Ising model [10–13], and the good associative memory of the Hopfield model defined on a network with large average connectivity [14]. In addition, the investigation of the metric structure [15–19] of these networks is of great interest. It was first empirically found [15] and then analytically derived [16,17] that scale-free networks are characterized by having a mean distance $\langle d\rangle$ between nodes scaling like the logarithm of the system size N, $\langle d\rangle\sim\ln(N)$.

The similarities between the structure of a BA network and a traditional Cayley tree have been recently studied. It has been found that they share many similarities. In fact, they are both generated by the subsequent addition of the same elementary unit (a node connected to m links) attached to the rest of the network in the direct or reverse direction [20]. The symmetry between these two types of networks is evident if we assume that each node i has an innate quality or ''energy'' ϵ_i and that the dynamics is parametrized by a variable T called the temperature that introduces a ''thermal noise,'' as has been done in self-organized models [21–23]. It is possible then to observe that the BA model becomes a limiting case of a scale-free network described by a Bose distribution of the energies to which the incoming links point [24], while the growing Cayley-tree network is described by a Fermi distribution of the energies at the interface [25]. Quantum networks (the bosonic scale-free network and the fermionic Cayley-tree network) evolve around a well defined core of initial nodes.

In this paper we focus our attention on the metric structure of quantum networks and in particular on their size, i.e., we estimate the distance measured over directed paths of a generic node i from the origin of the network and its dependence on the time t_i at which the node i was added to the network. We derive an expression for the average value of the distance $\langle \ell(t_i)\rangle$ from the origin of node i introduced at time t_i, measured on directed paths, in quantum networks at $T=\infty$. We find, in agreement with [15–17], that $\langle \ell(t_i)\rangle \sim\ln(t_i)$ in the bosonic scale-free network and we show for $T=\infty$ a similar behavior in fermionic networks.

At different values of T the topology of the network changes drastically. For energy distribution functions $p(\epsilon)\to 0$ for $\epsilon\to 0$, the exponent γ of the power-law connectivity distribution $P(k)\sim k^{-\gamma}$ goes from $\gamma=3$ at $T=\infty$ to $\gamma=2$ for $T=0$, and there is a phase transition at a critical temperature T_c below which a finite fraction of all the links is connected to a single node. Assuming that the bosonic network is a reasonable model for growing scale-free networks, we have, for example, that the citation network [26] with $\gamma\sim 3$ would correspond to a $T=\infty$ dynamics while the incoming component of the World Wide Web with $\gamma_{in}=2.1$ [7,27] would correspond to a low temperature dynamics. As T decreases the behavior of $\langle \ell(t_i)\rangle$ also changes, and the distance of node i from the origin depends less strongly on the time t_i of its arrival. At sufficiently low temperature, $\ell(t_i)$ remains constant as a function of t_i. In contrast, in the fermionic network, at $T=0$, when the dynamics becomes extremal, the network evolves far away from the origin and the distance of a node i from the origin of the network grows as a power law of the time t_i of its arrival in the network.

II. DISTANCES FROM THE ORIGIN IN THE BOSONIC NETWORK

The bosonic network is a generalization of the well known BA network [7]. In this model, a new node with m links is added to the network at each time step. Each node i has an innate quality or ''energy'' ϵ_i extracted from a probability distribution $p(\epsilon)$. The way the new links are attached follows a generalized preferential attachment rule: the probability Π_i that an existing node i acquires a new link depends on both its connectivity $k_i(t)$ and its energy ϵ_i, i.e.,

$$\Pi_i = \frac{e^{-\beta\epsilon_i} k_i(t)}{\sum_s e^{-\beta\epsilon_s} k_s(t)}, \tag{1}$$

with the parameter $\beta = 1/T$ tuning the relevance of the energy ϵ_i with respect to the connectivity $k_i(t)$. This network displays a power-law connectivity distribution $P(k) \sim k^{-\gamma}$ with $\gamma \in [2,3]$ depending on the $p(\epsilon)$ distribution and the inverse temperature β [24].

In the $T = \infty$ limit $(\beta = 0)$, the network reduces to a scale-free BA network [7] with an average connectivity k_i of node i that grows in time as a power law with exponent 1/2:

$$k_i(t) = m\sqrt{\frac{t}{t_i}}, \tag{2}$$

where t_i is the time at which node i was added to the network. The probability $p_{i,j}$ that two nodes i and j are connected by a link can be calculated from Eq. (1). Taking into account that at each time m new links are added to the network, the probability $p_{i,j}$ is given by m times Eq. (1). After substituting Eq. (2) into Eq. (1), we obtain at $T = \infty$ $(\beta = 0)$

$$p_{i,j} = \frac{m}{2} \frac{1}{\sqrt{t_i t_j}}. \tag{3}$$

The number of directed paths of length ℓ connecting node i, introduced in the network at time t_i, to node i_0, belonging to the original core of the network $(t_{i_0} = 1)$, is given by the mean value of the number of paths connecting node i to i_0 and passing through the points $i_0, i_1, i_2, \ldots, i_{\ell-1}, i_\ell = i$ with $t_{n+1} > t_n$. Indicating each node by the time of its arrival in the network, the probability of any directed path is given by the product $\Pi_{n=1}^{\ell} p_{n-1,n}$ with $t_{n-1} < t_n$. In order to find $n_\ell(t_i)$ we should sum over all possible paths. Replacing the sum over the nodes with the integrals over t_n, we obtain for $n_\ell(t_i)$

$$n_\ell(t_i) = \int_1^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \cdots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} p_{0,1} p_{1,2} \cdots p_{\ell-1,\ell}. \tag{4}$$

Using $t_0 = 1$, $t_\ell = t_i$ and Eq. (3) valid in the $T = \infty$ limit, we obtain

$$n_\ell(t_i) = \left(\frac{m}{2}\right)^{\ell} \int_1^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \cdots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} \frac{1}{t_1} \frac{1}{t_2} \cdots \frac{1}{t_{\ell-1}} \frac{1}{\sqrt{t_i}}$$

$$= \frac{1}{(\ell-1)!} \left(\frac{m}{2} \ln(t_i)\right)^{\ell-1} \frac{m}{2\sqrt{t_i}}. \tag{5}$$

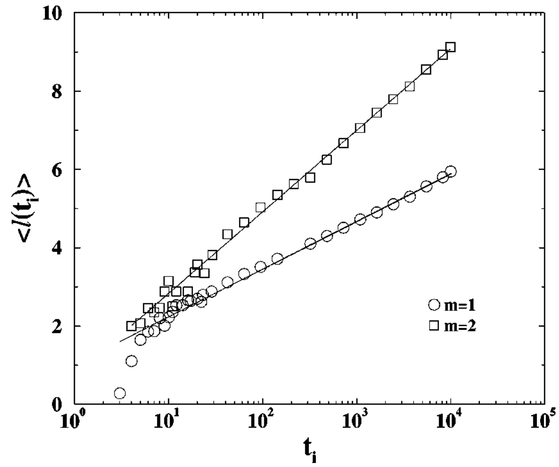

FIG. 1. Mean distance from the origin $\langle \ell(t_i) \rangle$ of the nodes arriving at time t_i in a BA network (a bosonic network at $T = \infty$) with $m = 1,2$. The solid lines indicate the theoretical prediction, Eq. (6).

This means that the mean distance between node i and a node i_0, such that $t_{i_0} = 1$, calculated only on directed paths, follows a Poisson distribution with average size

$$\langle \ell(t_i) \rangle = \frac{m}{2} \ln(t_i) \tag{6}$$

(see Fig. 1).

The distribution of the number of directed paths of length ℓ starting from the origin of the network is proportional to the integral of $n_\ell(t_i)$ over t_i,

$$P(\ell) \propto \int_1^t n_\ell(t') dt' = (-m)^\ell \left(1 - \frac{\Gamma(\ell, \ln(t)/2)}{\Gamma(\ell)}\right). \tag{7}$$

In Fig. 2 we show the agreement between the numerical results and Eq. (7) for a bosonic network of 10^4 nodes at $T = \infty$ and $m = 1,2$.

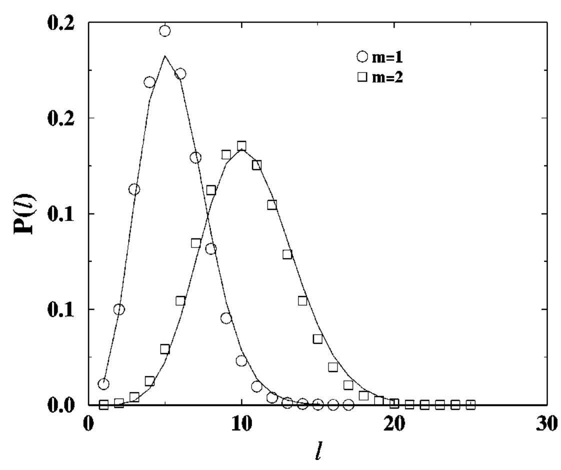

FIG. 2. Distribution of the number of directed paths of length ℓ in a BA network (a bosonic network at $T = \infty$) for $m = 1,2$. The solid lines are the analytical predictions, Eq. (7).

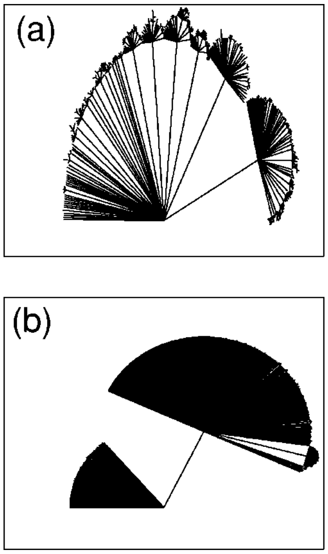

FIG. 3. Graphic representation of the bosonic network. (a) represents a network with $m=1$ and energy distribution given by Eq. (8) where $\theta=0.5$ at $T=\infty$ ($\beta=0$, the BA network), and (b) represents a network with the same parameters m and θ but with $T=1/\beta=0.33$ (network in the Bose-Einstein condensate phase). The number of nodes in both networks is $N=10^3$.

The topology of the bosonic network changes as a function of the temperature $T=1/\beta$. In particular, for a distribution $p(\epsilon)$ such that $p(\epsilon)\to 0$ as $\epsilon\to 0$, we know [24] that there is a critical temperature T_c below which the network has a topological transition and its structure is dominated by a single node that collects a finite fraction of all the links. For $T>T_c$ ($\beta<\beta_c$) the network is in the so-called "fit-gets-rich" (FGR) phase, while for $T<T_c$ ($\beta>\beta_c$) the network is in the so-called Bose-Einstein condensate (BEC) phase. To visualize this transition we have plotted the bosonic network with an energy distribution

$$p(\epsilon)=\frac{1}{\theta+1}\epsilon^{\theta}, \quad \epsilon\in(0,1), \tag{8}$$

where $\theta=0.5$, $m=1$, above [Fig. 3(a)] and below [Fig. 3(b)] the phase transition.

The network has been designed in order to underline its hierarchical structure. Starting from the single node at the origin of the tree, we have placed all the nodes that are directly attached to it on a semicircle of unitary radius, each node i separated from the next one by an angle $\Delta\alpha_i$ proportional to its connectivity, i.e.,

$$\Delta\alpha_i=\frac{k_i}{\sum_{j\in N(i)} k_j}, \tag{9}$$

where $N(i)$ are the nearest neighbors of node i added at a time $t_j>t_i$. We have repeated the same construction for all the nodes of the network in such a way that all the nearest neighbors of node i are on a semicircle of radius r_i with

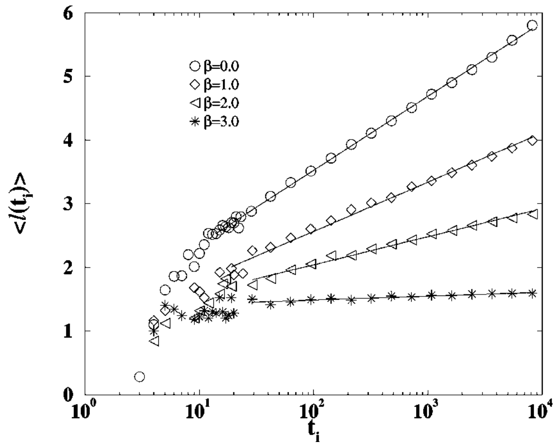

FIG. 4. Distances from the origin in a bosonic network with $m=2$ and $\theta=0.5$ as a function of β.

$$r_i=r_k\frac{k_i}{\sum_{j\in N(i)} k_j}, \tag{10}$$

where k is the node to which the node i was attached at time t_i. From Fig. 3 the change in the topology of the network at the critical temperature is clear, with the emergence of a single node that collects a finite fraction of all the links in the Bose-Einstein condensate phase.

As the topology of the network changes, the behavior of $\ell(i)$ as a function of t_i changes too. In fact, we have

$$\langle\ell(i)\rangle=a(\beta)\ln(t_i), \tag{11}$$

where the coefficient $a(\beta)$ is a decreasing function of the inverse temperature $\beta=1/T$.

In Fig. 4 we report $\langle\ell(i)\rangle$ for a bosonic network with $p(\epsilon)$ of the type (8) with $\theta=0.5$ and $m=2$ at different values of the inverse temperature β, above and below the critical value $\beta_c=1.7$ [24]. In order to illustrate the change in the topology of the network above and below the critical temperature, in Fig. 5 we report the behavior of different relevant structural quantities for a network of size $N=10^4$, $\theta=0.5$, and with $m=2$ around the critical inverse temperature $\beta_c=1.7$. We report the fraction of links attached to the most connected node, the exponent of the power-law component of the connectivity distribution, the clustering coefficient, and the coefficient $a(\beta)$. The fraction of links attached to the most connected network $k_{max}(\beta)/N$ is the order parameter of the FGR-BEC phase transition and increases as a function of β. The data reported in Fig. 5 are averaged over 100 runs. The connectivity distribution of the bosonic network contains a power-law component plus a point indicating the condensation phenomenon that appears for $\beta>\beta_c$. In Fig. 5 we report the exponent $\gamma(\beta)$ of the power-law component of the connectivity distribution, which decreases as a function of β, with an asymptotic value of $\gamma=2$. The data reported in Fig. 5 are averaged over 100 runs. The clustering coefficient $C(\beta)$ increases at the transition point while the coefficient $a(\beta)$ slowly decreases, saturating toward a zero value for

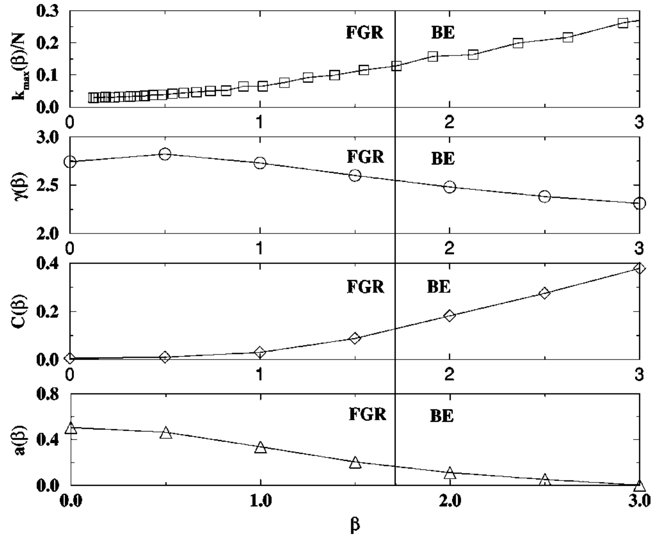

FIG. 5. Relevant structural quantities in a bosonic network of size $N=10^4$ with $m=2$ and $\theta=0.5$ as a function of the inverse temperature β.

$\beta>3.0$. The data reported in Fig. 5 for these last two quantities are averaged over 10 runs.

III. DISTANCES FROM THE ORIGIN IN THE FERMIONIC NETWORK

The fermionic network [25] is a growing Cayley tree, where the innate qualities of the nodes (energies) define their different branching tendencies. Starting at time $t=1$ from a node i_0 at the origin of the network, the node i_0 at time $t=2$ grows and m new nodes are directly connected to it. Each node i has an energy ϵ_i extracted from a given $p(\epsilon)$ distribution. At each time step a new node with connectivity 1 (at the interface) is chosen to branch, giving rise to m new nodes.

We assume that nodes with higher energy are more likely to grow than lower energy ones. In particular we take Π_i, the probability that a node i of the interface (with energy ϵ_i) grows at time t, to be

$$\Pi_i=\frac{e^{\beta\epsilon_i}}{\sum_{j\in \mathrm{Int(t)}} e^{\beta\epsilon_j}}, \tag{12}$$

where the sum in the denominator is extended to all nodes j at the interface Int(t) at time t. The model depends on the inverse temperature $\beta=1/T$. In the $\beta\rightarrow 0$ limit, high and low energy nodes grow with equal probabilities, and the model reduces to the *Eden model*. In the $\beta\rightarrow\infty$ limit the dynamics becomes extremal and only the nodes with the highest energy value are allowed to grow. In this case the model reduces to *invasion percolation* [28,29] on a Cayley tree.

Let us assume that $\beta=0$ $(T=\infty)$. The probability Π_i that a node i of the interface Int(t) grows at time t is given by

$$\Pi_i=\frac{1}{N_{Int}(t)}, \tag{13}$$

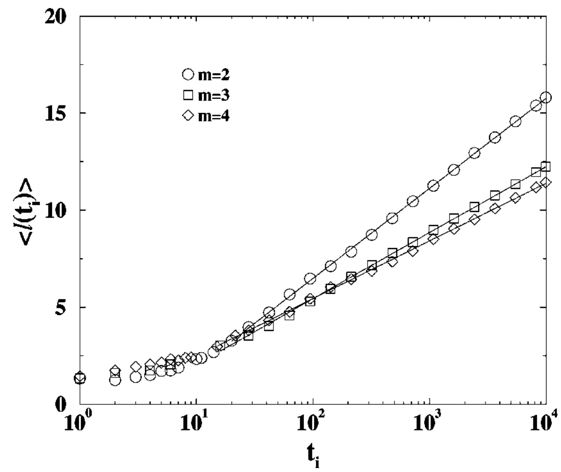

FIG. 6. Distance from the origin in a fermionic network at $T=\infty$ $(\beta=0)$ for networks of $N=10^4$ nodes with $p(\epsilon)$ uniform between zero and 1 and with $m=2,3,4$. The solid lines are the theoretical predictions of Eq. (20).

where $N_{Int}(t)$ is the total number of active nodes. Since at each time step a node of the interface branches and m new nodes are generated, after t time steps the model generates an interface of $N_{Int}(t)$ nodes, with

$$N_{Int}(t)=(m-1)t+1. \tag{14}$$

Let us denote by $\rho(t,t_i)$ the probability that a node created at time t_i is still at the interface at time t. Since every node i of the network grows with probability Π_i [Eq. (13)] if it is at the interface, in the mean field $\rho(t,t_i)$ follows as

$$\frac{\partial\rho(t,t_i)}{\partial t}=-\frac{\rho(t,t_i)}{N_{Int}(t)}. \tag{15}$$

Replacing Eq. (14) in Eq. (15) in the limit $t\rightarrow\infty$, we get the solution

$$\rho(t,t_i)=\left(\frac{t_i}{t}\right)^{1/(m-1)}. \tag{16}$$

Consequently, each node i that arrives at the interface at time t_i remains at the interface with a probability that decreases in time as a power law. The probability $p_{i,j}$ that a node i is attached to a node j (arriving in the network at a later time $t_j>t_i$) is given by the right-hand side of Eq. (13) calculated at time t_j. Taking into account Eq. (16) and the rate m of the addition of new nodes, we obtain for $p_{i,j}$,

$$p_{i,j}=\frac{m}{(m-1)t_j}\left(\frac{t_i}{t_j}\right)^{1/(m-1)}. \tag{17}$$

The number $n_\ell(t_i)$ of paths of length ℓ that connect a node i, introduced at time t_i, to the origin i_0 is given by the average number of paths connecting a node i to a node i_0 and passing

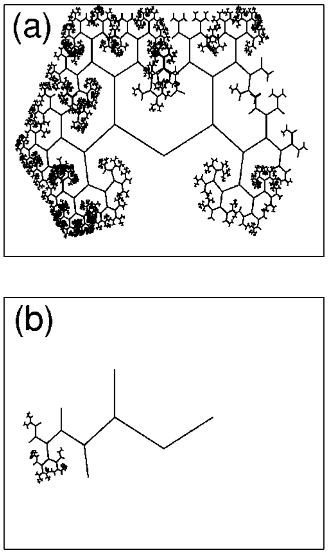

FIG. 7. The fermionic network with $m=2$ at $T=\infty$ (a) and at temperature $T=0.05$ (b). The number of nodes in both networks is $N=10^4$.

through the points $i_0,i_1,i_2,\ldots,i_{\ell-1},i_\ell=i$ with $t_{n+1}>t_n$. Indicating each node by the time of its arrival in the network and the sum over the nodes by the integrals over t_n, we obtain for $n_\ell(t_i)$

$$n_\ell(t_i)=\int_1^{t_i}dt_1\int_{t_1}^{t_i}dt_2\cdots\int_{t_{\ell-2}}^{t_i}dt_{\ell-1}p_{01}p_{1,2}\cdots p_{\ell-1,\ell} \tag{18}$$

and, using Eq. (17), we obtain, with a calculation analogous to Eq. (5),

$$n_\ell(t_i)\frac{1}{(\ell-1)!}\left(\frac{m}{m-1}\ln(t_i)\right)^{\ell-1}\left(\frac{1}{t_i}\right)^{1/(m-1)}. \tag{19}$$

This means that the mean distance between a node t_i and the origin follows a Poisson distribution with average size

$$\langle\ell(t_i)\rangle=\frac{m}{m-1}\ln(t_i). \tag{20}$$

As in the bosonic network, in the fermionic network at infinite temperature ($\beta=0$) the distance $\langle\ell(t_i)\rangle$ of node i from the origin grows logarithmically with the time t_i.

In Fig. 6 we report the analytical simulation of a fermionic network with $p(\epsilon)$ uniform between zero and 1, at $T=\infty$ and for $m=2,3,4$.

As the temperature decreases, the topology of the network changes drastically. In Fig. 7 we show the Cayley tree with $p(\epsilon)=1$, $\epsilon\in(0,1)$, and $m=2$ at infinite temperature ($\beta=0.0$) and at low temperature ($\beta=20.0$). At high temperature the network grows homogeneously in each direction, while at low temperature it evolves following only a single branch of the tree.

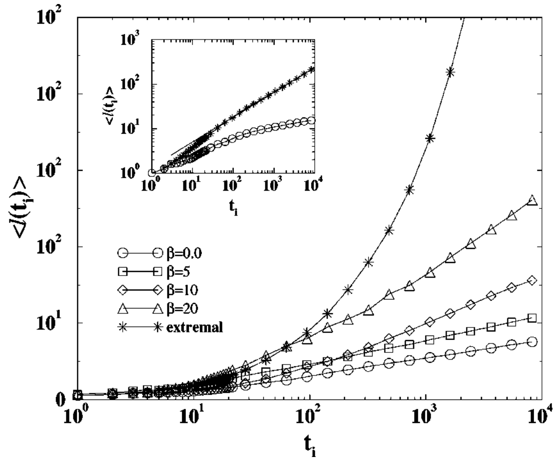

FIG. 8. Distance from the origin in a fermionic network with $m=2$ and $p(\epsilon)$ uniformly distributed between zero and 1 at different temperatures. At $\beta=0$ $(T=\infty)$ we have the predicted logarithmic behavior Eq. (20), while in the extremal case $\beta=\infty$ $(T=0)$ the network grows as a power law of the network size. The solid line in the inset is the power-law fit, Eq. (21), with $\zeta=0.55\pm0.05$.

The distance of a node i from the origin of the networks grows logarithmically with t_i at $T=\infty$ $(\beta=0)$. As the temperature decreases the behavior of $\langle\ell(t_i)\rangle$ gets steeper. For $p(\epsilon)$ uniformly distributed between zero and 1, in the extremal case $T=0$ $(\beta=\infty)$ when the node of highest energy grows deterministically at each time step, we have a dramatic change in the behavior, and $\langle\ell(i)\rangle$ grows as a power law of t_i:

$$\langle\ell(i)\rangle\propto(t_i)^{\zeta}. \tag{21}$$

$\zeta=0.55\pm0.05$ from the numerical results reported in Fig. 8.

IV. CONCLUSION

In conclusion, we have shown that bosonic and fermionic network are not only symmetrically built [20] but also at $T=\infty$ they are characterized by a distance $\langle\ell(t_i)\rangle$ from the origin that grows like the logarithm of the time t_i. In contrast, in the limit $T=0$ they behave in opposite ways: the bosonic network stays highly clusterized with a distance from the origin that remains constant as the network evolves, but in the fermionic network the distance $\langle\ell(t_i)\rangle$ grows like a power law of the time t_i.

ACKNOWLEDGMENTS

We are grateful to A. Capocci, P. Laureti, and Y.-C. Zhang for useful comments and discussions. This work was financially supported by the Swiss National Foundation under Grant No. 2051-067733.02/1 and by the European Commission Fet Open Project No. COSIN IST-2001-33555.

[1] L.A.N. Amaral, A. Scala, M. Barthélemy, and H.E. Stanley, Proc. Natl. Acad. Sci. U.S.A. **97**, 11 149 (2000).
[2] K.-I. Goh, E. Oh, H. Jeong, B. Kahng, and D. Kim, Proc. Natl. Acad. Sci. U.S.A. **99**, 12 583 (2002).
[3] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[4] S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[5] S. Strogatz, Nature (London) **410**, 268 (2001).
[6] G. Caldarelli, A. Capocci, P. De Los Rios, and M.A. Munõz, Phys. Rev. Lett. **89**, 258702 (2002).
[7] A.-L. Barabási and R. Albert, Science (Washington, DC, U.S.) **286**, 509 (1999).
[8] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[9] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).
[10] A. Aleksiejuk, J.A. Holyst, and D. Stauffer, Physica A **310**, 260 (2002).
[11] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, Phys. Rev. E **66**, 016104 (2002).
[12] M. Leone, A. Vazquez, A. Vespignani, and R. Zecchina, Eur. Phys. J. B **28**, 191 (2002).
[13] G. Bianconi, Phys. Lett. A **303**, 166 (2002).
[14] D. Stauffer, A. Aharony, L. da Fontoura, and J. Adler, e-print cond-mat/0212601.
[15] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[16] M.E.J. Newman, S.H. Strogatz, and D.J. Watts, Phys. Rev. E **64**, 026118 (2001).
[17] S.N. Dorogovtsev, J.F.F. Mendes, and A.N. Samukhin, e-print cond-mat/0210085.
[18] S. Wuchty and P. Stadler, http://www.santafe.edu/sfi/publications/Working-Papers/02-09-052.ps.gz
[19] R. Cohen and S. Havlin, Phys. Rev. Lett. **90**, 058701 (2003).
[20] G. Bianconi, Phys. Rev. E **66**, 056123 (2002).
[21] M. Vergeles, Phys. Rev. Lett. **75**, 1969 (1995).
[22] G. Caldarelli, A. Maritan, and M. Vendruscolo, Europhys. Lett. **35**, 481 (1996).
[23] M. Vergeles, Phys. Rev. E **55**, 6264 (1997).
[24] G. Bianconi and A.-L. Barabási, Phys. Rev. Lett. **86**, 5632 (2001).
[25] G. Bianconi, Phys. Rev. E **66**, 036116 (2002).
[26] S. Redner, Eur. Phys. J. B **4**, 131 (1998).
[27] A. Broder *et al.* Comput. Netw. **33**, 309 (2000).
[28] D. Wilkinson and J.F. Wiesenfeld, J. Phys. A **16**, 3365 (1983).
[29] N. Vanderwalle and M. Ausloos, Europhys. Lett. **37**, 1 (1997).